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TOPOLOGICAL, AFFINE AND ISOMETRIC ACTIONS ON FLAT RIEMANNIAN MANIFOLDS

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Dedicated to the memory of Noel J. Hicks

Let M be a closed Riemannian flat manifold. It is well-known that Out $\pi_1(M)$, the outer automorphism group of the fundamental group of M, is isomorphic to the group $\pi_0(\mathcal{E}(M))$ of homotopy classes of self homotopy equivalences of M.

A homomorphism $\varphi\colon G \to \operatorname{Out} \pi_1(M) \cong \pi_0(\mathcal{E}(M))$ is called an *abstract kernel* and is denoted by $(G, \pi_1(M), \varphi)$. A *geometric realization* of $(G, \pi_1(M), \varphi)$ *by a group of homeomorphisms* is a homomorphism $\hat{\varphi}\colon G \to \mathcal{H}(M)$, where $\mathcal{H}(M)$ is the group of homeomorphisms of M, so that $\hat{\varphi}$ composed with the natural homomorphism $\mathcal{H}(M) \to \operatorname{Out} \pi_1(M)$ agrees with φ. This paper is concerned with the geometric realization problem when G is *finite* and M is *flat*, and is related to some of the ideas promulgated in [7].

In order that an abstract kernel has a geometric realization the kernel must have an "algebraic realization," [2, 2.2]. The Corollary to Lemma 1 characterizes the type of group extension which must exist if one is to find a geometric realization by an *effective* group of homeomorphisms on a closed aspherical manifold. Then Theorem 3 asserts that this necessary condition is also sufficient for an effective geometric realization on Riemannian flat manifolds. Because of flatness this realization can always be chosen to be a group of *affine* diffeomorphisms which, as we show in Theorems 3 and 6, is affinely equivalent to an isometric action on an affinely equivalent flat manifold. Thus it will follow that the finite groups which act effectively on M are isomorphic to those groups which act *isometrically* on manifolds *affinely equivalent* to M.

If, on the other hand, one is willing to sacrifice effectiveness one needs, as shown in the Corollary to Theorem 4, only the existence of *some* group

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extension realizing the abstract kernel to geometrically realize an abstract kernel. But one can construct many examples of abstract kernels which fail to have geometric realizations because of the failure of the existence of an algebraic extension (this can only by done when the center of $\pi_1(M)$ is not trivial). However, we do show (Theorem 5) that for any abstract kernel $(G, \pi_1(M), \varphi)$, one can find an epimorphism $H \to G$ of a finite group H so that the composition $H \to G \xrightarrow{\varphi} \mathrm{Out}\, \pi_1(M)$ always admits a geometric realization by an affine group of diffeomorphisms. A result of this type seems to us to be the most natural way of attempting to solve the realization problem. In fact, when $\pi_1(M)$ has a nontrivial center and the necessary algebraic extensions fail to exist, this is the only possible avenue left for a positive result.

Let π be a discrete group with torsion free center. An extension E of π by a group G is said to be *admissible* if in the induced diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\bar\varphi} & & \downarrow{\varphi} & & \\ 1 & \longrightarrow & \mathrm{Inn}\,\pi & \longrightarrow & \mathrm{Aut}\,\pi & \longrightarrow & \mathrm{Out}\,\pi & \longrightarrow & 1 \end{array}$$

$\bar\varphi$ is injective on any finite subgroup of E. Of course, the automorphism $\bar\varphi(e)$ of π is conjugation by e, $e \in E$.

Lemma 1. *Let M be closed aspherical manifold with $\pi_1(M) = \pi$. Then the extension $1 \to \pi \to N_{\mathcal{H}(\tilde M)}(\pi) \xrightarrow{\eta} \mathcal{H}(M) \to 1$ is admissible, where $\mathcal{H}(M)$ is the group of all self homeomorphisms of M, $\tilde M$ is the universal cover of M, and $N_{\mathcal{H}(\tilde M)}(\pi)$ denotes the normalizer of π in $\mathcal{H}(\tilde M)$.*

Proof. Suppose that there is $z \in N_{\mathcal{H}(\tilde M)}(\pi)$ with finite order, and $\bar\varphi(z) = 1$. Let F be the finite cyclic subgroup generated by z. Consider the induced extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & \eta^{-1}(\eta(F)) & \xrightarrow{\eta} & \eta(F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}\,\pi & \longrightarrow & \mathrm{Aut}\,\pi & \longrightarrow & \mathrm{Out}\,\pi & \longrightarrow & 1. \end{array}$$

Since F is finite and π is torsion-free, η is an isomorphism on F so that we have a semi-direct product structure on $\eta^{-1}(\eta(F))$. Now since $z \in \ker \bar\varphi$, conjugation by z yields $zxz^{-1} = x$ for all $x \in \pi$. This implies $\eta^{-1}(\eta(F)) = \pi \times F$. So the action $(\eta^{-1}(\eta(F)), \tilde M)$ contains a finite subgroup action $(F, \tilde M)$ which commutes with $(\pi, \tilde M)$ so that $\bar\varphi(F) = 1$ in $\mathrm{Aut}\,\pi$. This implies by [2, A.11] that the action of F on M must be trivial. Thus we have $z = 1$.

Corollary. *Let (G, M) be an effective action of a finite group on a closed aspherical manifold M with $\pi_1(M) = \pi$. Then the induced extension $1 \to \pi \to E$*

$\to G \to 1$, *where* E *denotes the group of all liftings of* G *to homeomorphisms of* $\tilde{M}$, *is admissible.*

Notice that $E = \eta^{-1}(G)$ and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & N_{\mathcal{H}(\tilde{M})}(\pi) & \longrightarrow & \mathcal{H}(M) & \longrightarrow & 1 \\ & & \updownarrow = & & \uparrow \cup & & \uparrow \cup & & \\ 1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

commutes. The action of E on $\tilde{M}$ can be constructed explicitly as done in [2, 2.2].

Remark. Let (G, M) be an action (not necessarily effective) of a finite group on a closed aspherical manifold M with $\pi_1(M) = \pi$. Then there exists an extension (not necessarily admissible) $1 \to \pi \to E \to G \to 1$ realizing the abstract kernel φ: $G \xrightarrow{\hat{\varphi}} \mathcal{H}(M) \xrightarrow{\varphi'} \text{Out}\ \pi_1(M)$.

Proof. Since $(\hat{\varphi}(G), M)$ is effective, there exists an admissible extension E' of π by $\hat{\varphi}(G)$, $1 \to \pi \to E' \to \hat{\varphi}(G) \to 1$. We can "pull-back" $G \xrightarrow{\hat{\varphi}} \hat{\varphi}(G)$ along $E' \to \hat{\varphi}(G)$ to get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow = & & \vdots & & \downarrow \hat{\varphi} & & \\ 2 & \longrightarrow & \pi & \longrightarrow & E' & \longrightarrow & \hat{\varphi}(G) & \longrightarrow & 1. \end{array}$$

Certainly the top row is an extension of π by G realizing $(\pi, G, \varphi = \varphi' \circ \hat{\varphi})$.

We recall the definitions of abstract crystallographic and Bieberbach groups as we shall use them. An *abstract crystallographic group of rank* n is any group which is isomorphic to a uniform discrete subgroup of the Euclidean group $E(n)$ of motions on $\mathbf{R}^n$. An *abstract Bieberbach group of dimension* n is any torsion free crystallographic group of rank n. The classical Bieberbach theorems *characterize* these *intrinsically* by: E is an abstract crystallographic group of rank n if and only if it contains a normal free abelian group of rank n of finite index which is maximal abelian. E is an abstract Bieberbach group of dimension n if and only if it is a torsion free crystallographic group of rank n. In both cases the finite quotient group acts *faithfully* on $\mathbf{Z}^n$. The quotient group is called the *holonomy* group when E is torsion free. We refer the reader to [9, Chapter 3] for further general details.

Proposition 2. *An admissible extension of an abstract crystallographic group by a finite group is an abstract crystallographic group.*

Proof. Let E be an admissible extension of a crystallographic group π by a finite group G so that we have short exact sequence $1 \to \pi \to E \to G \to 1$. π

has a torsion-free, maximal abelian, normal subgroup $\mathbf{Z}^n$ of finite index. We shall show that $C_E(\mathbf{Z}^n)$, the centralizer of $\mathbf{Z}^n$ in E, is torsion-free, normal, maximal abelian and of finite index in E.

We check that it is torsion-free. Let $Z^1(\pi/\mathbf{Z}^n; \mathbf{Z}^n)$ be the group of automorphisms of π which induce the identity on $\mathbf{Z}^n$. Then we have a homomorphism $C_E(\mathbf{Z}^n) \to Z^1(\pi/\mathbf{Z}^n; \mathbf{Z}^n)$ (via conjugation) with kernel $C_E(\pi)$. But it is known that $Z^1(\pi/\mathbf{Z}^n; \mathbf{Z}^n)$ is the group of 1-cocycles of the "holonomy" $\pi/\mathbf{Z}^n$ in $\mathbf{Z}^n$ [1, Lemma 6] or [2, §6], and hence is torsion-free. This implies that $C_E(\pi)$ and $C_E(\mathbf{Z}^n)$ have the same torsion elements. Since $1 \to \pi \to E \to G \to 1$ is admissible, $C_E(\pi)$ is torsion-free (see Remark below) and hence so is $C_E(\mathbf{Z}^n)$.

We now claim that $C_E(\mathbf{Z}^n)$ is abelian. It is a torsion-free central extension of $\mathbf{Z}^n$ by a finite group $A = C_E(\mathbf{Z}^n)/\mathbf{Z}^n$. Consider the injective toral action which we could construct using $(T^n, T^n \times \text{point}, A) \to (\text{point } A)$. Since $C_E(\mathbf{Z}^n)$ is torsion-free and a central extension, T^n acts almost effectively on $(T^n \times \text{point})/A$, so it must be a torus, and hence $A \subset T^n$. So $C_E(\mathbf{Z}^n) \cong \pi_1(T^n/A) \cong \mathbf{Z}^n$.

The other facts are easily verified. This completes the proof of Proposition 2.

Remark. Here is another point of view of *admissibility*. Let π be any discrete group with $\mathcal{Z} = \mathcal{Z}(\pi)$, the center of π, torsion free and finitely generated of rank k, and G a finite group. *An extension of π by G*

$$1 \to \pi \to E \to G \to 1$$

is admissible if and only if $C_E(\pi)$ is torsion-free (and hence if and only if $C_E(\pi)$ is a free abelian group of rank k).

To obtain this fact one shows, by diagram chasing, the induced sequences in the diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Z}(\pi) & \longrightarrow & C_E(\pi) & \longrightarrow & \ker \varphi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \bar{\varphi}} & & \downarrow{\scriptstyle \varphi} & & \\
1 & \longrightarrow & \mathrm{Inn}\ \pi & \longrightarrow & \mathrm{Aut}\ \pi & \longrightarrow & \mathrm{Out}\ \pi & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

are exact. Now, $\bar{\varphi}|$(any torsion subgroup of E) is injective $\Leftrightarrow \ker \bar{\varphi} = C_E(\pi)$ is torsion-free. For the second statement we have, as in the proof of our preceding Proposition 2, that $C_E(\pi)$ must be free abelian of rank k. Note also that this means that kernel φ will be finite abelian and isomorphic to a subgroup of a k-torus.

Proposition 2 has an important application to geometric realization of finite groups of homotopy classes of self homotopy equivalences on flat manifolds.

Theorem 3. *Let $M(\pi)$ be a closed Riemannian flat manifold. If an abstract kernel (G, π, φ) admits an admissible extension E, then there is a geometric realization of this extension by an effective affine action of G on $M(\pi)$ which is affinely equivalent to an isometric action on an affinely equivalent flat manifold $M(\theta(\pi))$. Furthermore, the lifting of this affine action to $\tilde{M}(\pi)$ induce the same automorphisms of π as E.*

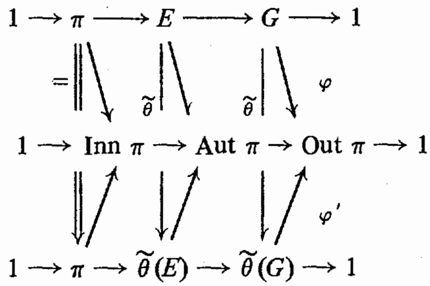
Proof. Let $1 \to \pi \to E \to G \to 1$ be a given admissible extension. By Proposition 2, E is an abstract crystallographic group. So we have an abstract isomorphism θ of E into $E(n)$, the group of rigid motions. Note that $\theta|_\pi$ is an isomorphism between two genuine Bieberbach groups π and $\theta(\pi)$. Therefore there exists $\tilde{h} \in A(n)$, the group of affine motions, such that $\tilde{h}\sigma\tilde{h}^{-1} = \theta(\sigma)$ for all $\sigma \in \pi$. Since $\theta(E)$ is crystallographic, we have the action of $\theta(E)/\theta(\pi) = \tilde{\theta}(G)$ on the flat Riemannian manifold $M(\theta(\pi)) = \mathbf{R}^n/\theta(\pi)$, as a group of isometries.

We define an affine diffeomorhism h: $M(\pi) \to M(\theta(\pi))$ coming from the affine map $\tilde{h}$: $\mathbf{R}^n \to \mathbf{R}^n$ induced from $\theta(\sigma) = \tilde{h}\sigma\tilde{h}^{-1}$. We consider the isometric action on $\mathbf{R}^n$ given by $\theta(E) \subset E(n)$, and define a new action of $\tilde{\theta}(E)$ on $\mathbf{R}^n$ by $\tilde{\theta}(e) = \tilde{h}^{-1} \circ \theta(e) \circ \tilde{h}$. This induces an affine action of $\tilde{\theta}(G)$ on $M(\pi)$, $\tilde{\theta}(g) = h^{-1} \circ \bar{\Theta}(g) \circ h$. Thus the action $\tilde{\theta}(G)$ on $M(\pi)$ is affinely equivalent to the isometric action of $\bar{\Theta}(G)$ on $M(\theta(\pi))$.

So we have a commutative diagram between two extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\tilde{\theta}} & & \downarrow{\scriptstyle\tilde{\theta}} & & \\ 1 & \longrightarrow & \pi & \longrightarrow & \tilde{\theta}(E) & \longrightarrow & \tilde{\theta}(G) & \longrightarrow & 1. \end{array}$$

Note that the extension $\alpha = \tilde{\theta}^*(\alpha')$, where $\tilde{\theta}^*$: $H^2_{\varphi\circ\tilde{\theta}^{-1}}(\tilde{\theta}(G), \mathfrak{Z}(\pi)) \to H^2_\varphi(G, \mathfrak{Z}(\pi))$, $\mathfrak{Z}(\pi)$ = Center of π. In the induced diagram



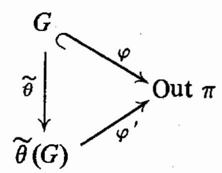
one can show that all triangles are commutative. This proves that the lifting of the new affine action to $\tilde{M}(\pi)$ induces the same automorphisms of π as E. q.e.d.

By Lemma 1, the existence of an admissible extension is necessary for an *effective* action. So Theorem 3 says that *this necessary condition is also sufficient.*

If the abstract kernel φ: $G \to \mathrm{Out}\,\pi$ is *injective*, then the admissibility condition is *automatically satisfied* provided that an extension *exists*. This means that the only obstruction to the realization of a group of homotopy classes of self homotopy equivalences on a flat manifold by a group of affine actions is simply the existence of an extension. Thus this gives us an algebraic solution to the finite realization problem.

Corollary. *Let M be a closed flat Riemannian manifold, H: $M \to M$ a homotopy equivalence with H^k homotopic to the identity with H^r not homotopic to the identity for $1 \leqslant r < k$. Then H is homotopic to an affine diffeomorphism K such that $K^k =$ identity if and only if the abstract kernel $\mathbf{Z}/k\mathbf{Z} \to$* Out $\pi_1(M)$ *arising from H admits an extension.*

Proof. Let $G = \{\Psi(H)|0 \leqslant i < k\}$ be the subgroup of Out π, where Ψ: $\mathfrak{E}(M) \to \mathrm{Out}(\pi)$. If $\mathbf{Z}/k\mathbf{Z} \cong G \hookrightarrow \mathrm{Out}\,\pi$ admits an extension, there then exists a subgroup $\tilde{\theta}(G)$ of Aff(M) isomorphic to G. Note that in the commutative diagram



φ' is really the composition $\tilde{\theta}(G) \hookrightarrow \mathrm{Aff}(M) \hookrightarrow \mathfrak{E}(M) \xrightarrow{\Psi} \mathrm{Out}\,\pi$. This implies that $K = \tilde{\theta}(\Psi(H))$ is homotopic to H and completes the proof.

The reader will surely wonder if the existence of a nonadmissible extension has something to do with the existence of an ineffective geometric realization. The corollary to our next theorem shows that this suspicion is correct.

We need two facts for the proof of Theorem 4.

Fact 1, [2, 2.2]. Let M be a path connected space admitting covering space theory. If an effective action (G, M) of a finite group on M is given, then the liftings E of G to $\tilde{M}$ induces a "unique" (in $H^2(G; \mathfrak{Z}(\pi))$) extension $1 \to \pi_1(M) \to E \to G \to 1$. We denote this extension by $l(G, M) \in H^2(G; \mathfrak{Z}(\pi_1(M)))$.

One obtains uniqueness after one specifies base points. An explicit expression for E is described in [2, 2.2]; also compare with the remark following Lemma 1.

Fact 2. Let $0 \to \mathbf{Z}^n \to C \xrightarrow{\nu} F \to 1$ be a central extension with F finite. Then C contains a characteristic finite subgroup L which contains all the torsion of C. Moreover, C/L is free abelian of rank n.

Proof of Fact 2. We know of no handy algebraic reference, although R. Griess has shown us an argument. On the other hand, Fact 2 does admit several interesting "geometric" proofs which also describe certain types of actions on the n-torus. This can be used, although we shall not do it here, to describe homotopically the actions on the n-torus of finite groups of homotopically trivial homeomorhisms. We shall employ the theory of *injective toral actions*. See [4] or [3]. This enables us to do the following.

Let (W, N) denote a properly discontinuous action of a discrete group N on a simply connected space W. For each central extension

$$a: \qquad 0 \to \mathbf{Z}^n \to C \to N \to 1, \quad a \in H^2(N; \mathbf{Z}^n),$$

we may associate on $T^n \times W$ an action of $T^n \times N$ with $T^n \times 1$ acting by translation on the first factor. The projection $T^n \times W \to W$ is equivariant where $T^n \times 1$ acts trivially on W. (Actually, associated with each extension there is a set of such actions all of which are $T^n \times N$ equivalent.) On the universal covering $R^n \times W$ one obtains an action of C with the $\mathbf{Z}^n$ in C just acting as translations on the first factor and with $R^n \times W/\mathbf{Z}^n$ yielding the constructed $(T^n \times N, T^n \times W)$.

For our purpose, choose $W =$ a point and for N take the F given in the hypothesis. We then obtain a $T^n \times F$ action on T^n, and we may lift the F action to a C action on $\mathbf{R}^n$. Let L be the subgroup of C which fixes all of $\mathbf{R}^n$. Certainly $\nu: L \to \nu(L) \subset F$ is a monomorphism, and so L is a finite (normal) subgroup.

We claim that L is the set of all elements of C having finite order. For if $e \in C$ has finite order, then $e \to e' \in C/L$ has finite order, and the cyclic group generated by e' acts effectively on $\mathbf{R}^n$. But if e' is not the identity, then some power of it is a cyclic p-group H for some prime p and H, by Smith theory, fixes a p-acyclic subset V of $\mathbf{R}^n$. Of course H, actually $\nu(H)$, fixes V^* in T^n. But as C is a central extension of $\mathbf{Z}^n$, [4, Corollary 6.2] implies that $\nu(H)$ acts trivially on T^n, and so H acts trivially on $\mathbf{R}^n$ forcing e to be in L. (The point is that $\nu(H)$ fixes $V^* \subset T^n$, and the complete lift of $\nu(H)$ yields H commuting with $\mathbf{Z}^n$ and both invariant on V, hence $V^* = T^n$ because $H^*(V^*: \mathbf{Z}_p) \approx H^*(\mathbf{Z}^n; \mathbf{Z}_p)$.)

It is clear now that the inessential part of the induced F action on T^n is $\nu(L)$. This yields an effective action of $F/\nu(L)$ on T^n which induces the effective action of $C/L = A$, from C on $\mathbf{R}^n$. Using the argument above, A *is torsion-free* and so must be an admissible central extension of $\mathbf{Z}^n$ by $F/\nu(L)$. Consequently, $F/\nu(L)$ acts freely on T^n and is imbedded in the T^n action since F acts trivially on $W =$ point. Therefore $T^n/(F(\nu(L)))$ is again a torus, and its fundamental group is our group A. This completes the proof of Fact 2.

For Theorem 4 we shall need M to be a reasonable type of space, a manifold say, whose fundamental group has *finitely generated torsion free center*. Let $\mathcal{A}(M)$ be a subgroup of $\mathcal{H}(M)$, the group of homeomorphism of M. G and H will refer to finite groups.

Theorem 4. *The following two statements are equivalent*:

(A) *If an abstract kernel* (π, G, φ) *has an admissible extension* α, *then* G *can be realized effectively as a subgroup of* $\mathcal{A}(M)$ *so that* $\alpha = l(G, M)$.

(B) *If an abstract kernel* (π, H, φ) *has an extension* α, *then* H *can be realized by* $H \xrightarrow{p} p(H) \subset \mathcal{A}(M)$ *so that* $\alpha = p^* l(p(H), M)$.

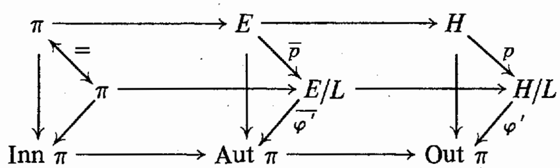
Proof. (A $\Rightarrow$ B) Suppose $1 \to \pi \to E \to H \to 1$ is an extension α realizing (π, H, φ). Then we have the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Z}(\pi) & \longrightarrow & C_E(\pi) & \longrightarrow & C/\mathcal{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Inn} \pi & \longrightarrow & \operatorname{Aut} \pi & \longrightarrow & \operatorname{Out} \pi & \longrightarrow & 1
\end{array}$$

Since the first row is a central extension and $C/\mathcal{Z}$ is finite, we may apply Fact 2 and obtain L, the torsion subgroup of $C_E(\pi)$, whose quotient $C_E(\pi)/L = A$ is abelian. We get

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & \xrightarrow[\nu]{=} & L & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{Z}(\pi) & \rightarrow & C_E(\pi) & \rightarrow & C/\mathcal{Z} & \rightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{Z}(\pi) & \rightarrow & A & \rightarrow & A/\mathcal{Z} & = \text{finite abelian} \rightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Note that L is characteristic in $C_E(\pi)$, and hence L is normal in E. Also note that L is in H. Thus we have a commutative diagram:



Since

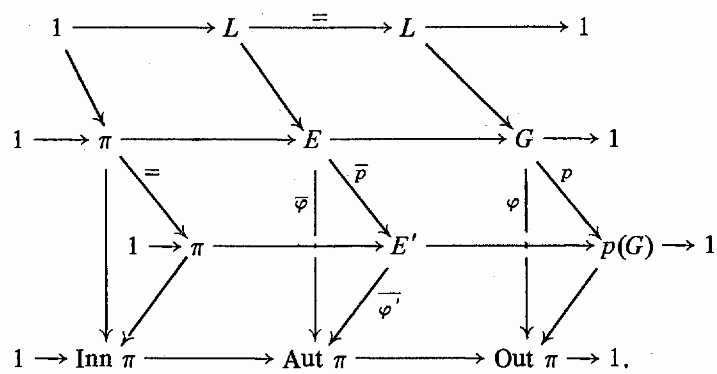
$$\ker \overline{\varphi'} = \bar{p}(\ker \bar{\varphi}) = C_E(\pi)/L = A,$$

$\ker \bar{\varphi}'$ is torsion-free. Then by (A), $H/L \xrightarrow{\varphi'} \text{Out}\,\pi$ can be realized effectively so that $1 \rightarrow \pi \rightarrow E/L \rightarrow H/L \rightarrow 1$ is the lifting of the effective action $(H/L, M)$ to $\tilde{M}$, i.e., $\alpha = p^*l(H/L, M) = p^*l(p(H), M)$, with $p(H)$ in $\mathcal{A}(M)$.

(B $\Rightarrow$ A) Let $1 \rightarrow \pi \rightarrow E \rightarrow G \rightarrow 1$ be an admissible extension α. By (B), there is a realization $p\colon G \rightarrow p(G) \subset \mathcal{A}(M) \subset \mathcal{H}(M)$ with $l(p(G), M) = 1 \rightarrow \pi \rightarrow E' \rightarrow p(G) \rightarrow 1$, and α-$p^*l(p(H), M)$. Note the last condition means simply that we have a commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \pi & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
 & & \downarrow{=} & & \downarrow{\bar{p}} & & \downarrow{p} & & \\
1 & \rightarrow & \pi & \rightarrow & E' & \rightarrow & p(G) & \rightarrow & 1
\end{array}
$$

Let L = kernel of $\bar{p}$. Then



Now $L = \text{kernel}\,\bar{p} \subset \text{kernel}\,\overline{\varphi'} \circ \bar{p} = \text{kernel}\,\bar{\varphi}$, so L is torsion free. Since $L < G$, L is finite. Thus $L = \{1\}$, which implies p is 1-1. Therefore $G \xrightarrow{\varphi} \text{Out}\,\pi$ is realized as a group of homeomorphisms $(G, M) = (p(G), M)$, (in $\mathcal{C}(M)$), so that $l(G, M) = \alpha$, which is what we wanted to prove.

Remark. An arbitrary manifold is not likely to satisfy condition (A) or (B). However one might expect this to hold for closed aspherical manifolds. In fact, it is known to hold for closed surfaces and, in increasing generality, it holds for complete, finite volume hyperbolic manifolds (dimension > 2), certain classes of locally symmetric spaces and the manifolds described in [7]. Theorem 3 states that $M(\pi)$, a closed Riemannian flat manifold, satisfies condition (A). Explicitly we have

Corollary. *Let $1 \to \pi \to E \to G \to 1$ be an extension of the fundamental group π of a flat manifold $M(\pi)$ realizing the abstract kernel (G, π, φ). Then there exists a geometric realization of this kernel by an affine action of G on $M(\pi)$. The subgroup L of G, which acts ineffectively on $M(\pi)$, splits to a normal subgroup of E whose quotient E/L is an admissible extension realizing the induced abstract kernel $(G/L, \pi, \varphi)$.*

It is interesting to relate this corollary to Proposition 2. It implies, in particular, that any extension E of a Bieberbach group by a finite group contains a normal finite group L whose quotient E/L is an abstract crystallographic group.

At this point we shall sketch quite a different approach to Theorem 3 and the Corollary to Theorem 4. From the hypothesis we construct the diagram of extensions:

$$\begin{array}{cccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
i^*b = a\colon & 1 & \longrightarrow & \mathbf{Z}^n & \longrightarrow \pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
\uparrow{\scriptstyle i^*} & & & \downarrow{\scriptstyle =} & \downarrow & & \downarrow{\scriptstyle i} & & \\
b\colon & 1 & \longrightarrow & \mathbf{Z}^n & \longrightarrow E & \longrightarrow & E/\mathbf{Z}^n & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & \xrightarrow{=} & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Here we think of a and b as cohomology classes in $H^2_\gamma(E/\mathbf{Z}^n; \mathbf{Z}^n)$ and $H^2_\gamma(\Phi; \mathbf{Z}^n)$ representing these extensions. Now the theory of *injective Siefert fiberings* (which is a refinement of the theory of injective toral actions used in the proof of Fact 2), allows us to construct for each 1-cocycle in $Z^1_\gamma(E/\mathbf{Z}^n; T^n)$ and $Z^1_\gamma(\Phi; T^n)$ a special type of action of $E/\mathbf{Z}^n$ and Φ on the n-torus T^n. Two actions will be equivalent if and only if the cocycles are cohomologous. Then via the compatible isomorphisms δ: $H^1_\gamma(E/\mathbf{Z}^n; T^n) \overset{\cong}{\to} H^2_\gamma(E/\mathbf{Z}^n; \mathbf{Z}^n)$ and δ: $H^1_\gamma(\Phi; T^n) \overset{\cong}{\to} H^2_\gamma(\Phi; \mathbf{Z}^n)$ the extension class represents the class of equivalent actions. (Moreover the lift of this action to the universal covering can be described by a 2-cocycle representing this extension class, and moreover this particular extension is the group which is acting on the universal covering.) See pages 70 and 82 of [3] where one should take $N = E/\mathbf{Z}^n$ and $W =$ a point. Now the constructed action on T^n, in this special case at hand, is given cohomologically by a crossed-homomorphism which translates geometrically into an affine action on a flat torus T^n. Thus one constructs first on T^n the affine $E/\mathbf{Z}^n$ action according to the class b. Since b pulls back to $a = i^*(b)$, the restricted Φ-action on T^n yields $M(\pi)$, and, with care, one can do this part isometrically. Then on $M(\pi)$ one produces the desired affine $G = (E/\mathbf{Z}^n)\Phi$ action. The reader will find further exploitation of this particular technique in [3, p. 83] where one even obtains the existence of the desired extensions for certain subgroups of Out $\pi_1(M)$.

We caution the reader of the preceding corollary that from a group action on M we get an abstract kernel. From this abstract kernel we may create various extensions. These extensions are not unique (the extensions are in 1-1 correspondence with the elements of $H^2_\varphi(G; \mathfrak{Z}(\pi))$. By the corollary a created extension admits a geometric realization by an affine group of diffeomorphisms. However, while the extension corresponding to lifting the original group action to $\tilde{M}$ and the created extension both represent the same abstract kernel, they may be very different extensions and may lead to very different affine realizations. The differences in the affine actions, as we shall see in Theorem 5, are essentially manifested in the contributions coming from the connected component of the identity of the group of isometries of $M(\pi)$, which is a torus of rank k (k = rank of center of π).

In Theorems 3 and 4 and their corollaries we have assumed that the abstract kernel admits an extension E. We then showed that this particular extension can be realized as the lifting of an affine action of G on M to its universal covering $\tilde{M}$. However, as is known, an abstract kernel (G, π, φ) may not admit any extension of π by G (see [5] or a modification of the examples in [8]), and consequently the finite group of homotopy classes of self-homotopy equivalences $\Psi^{-1}(\varphi(G))$, where Ψ: $\pi_0(\mathcal{E}(M)) \overset{=}{\to} \text{Out}\ \pi_1(M)$, cannot be geometrically realized by a group of homeomorphisms. However, in our next theorem, we shall show that G can be "enlarged" to a group H whose resulting abstract kernel admits a geometric realization by a group of affine diffeomorphisms.

Theorem 5. *Let (G, π, φ) be an abstract kernel of a finite group G with $M(\pi)$ a closed flat manifold. Then there exist a finite group H and an epimorphism μ_G: $H \to G$ such that the abstract kernel φ_H: $H \overset{\mu_G}{\to} G \overset{\varphi}{\to} \text{Out}\ \pi$ can be geometrically realized by a group of affine diffeomorphisms. Moreover H can be chosen so that the kernel μ_G is a subgroup, which depends only on M and not on G, of the connected component of the isometries of M.*

Proof. Let k = dimension of $H_1(M; \mathbf{Q})$. We use the following result of [5]:

There exists an exact sequence

$$0 \to \Lambda_1/\Lambda \to A(\pi) \overset{\mu}{\to} \text{Out}\ \pi \to 1$$

where Λ_1/Λ is a *finite* subgroup of $T^k = \text{Aff}_0(M)$, the connected component of the group of isometries of M, $A(\pi) < \text{Aff}(M)$, and μ is the restriction of the natural homomorphism $\text{Aff}(M) \to \text{Out}\ \pi$. In fact, $\Lambda_1 \cong H_1(M; \mathbf{Z})/\text{Torsion}$ and $\Lambda \cong \mathfrak{Z}(\pi)$, and the sequence

$$0 \to \Lambda \to H_1(M; \mathbf{Z})/\text{Torsion} \to \Lambda_1/\Lambda \to 0$$

is exact.

Put $\bar{G} = \varphi(G)$ and $A = \mu^{-1}(\bar{G})$. We form the "pull-back" $H \subset A \times G$ by $H = \{(a, g) |\ \mu(a) = \varphi(g)\}$. This group projects *onto* both A and G with

$$\text{kernel}\ \mu_A = 1 \times \text{kernel}\ \varphi, \quad \text{kernel}\ \mu_G = \text{kernel}\ \mu \times 1.$$

The group H now acts, via μ_A: $H \to A$, as a group of affine diffeomorphisms on the flat manifold $M(\pi)$. This action may not be effective. The subgroup of H which fixes M is precisely isomorphic to the kernel of φ. Note that $\Lambda_1/\Lambda \subseteq T^k$ will act as isometries on $M(\pi)$. In fact, in the Calabi fibration, [9, 3.63], $M_{n-k} \to M \to T^k = \mathbf{R}^k/\Lambda_1$, Λ_1/Λ moves only along the fibers.

Remark. It seems reasonable to *conjecture* that for any closed aspherical manifold a homomorphism $G \overset{\varphi}{\to} \text{Out}\,\pi$ can always be "enlarged" to group H so that $H \overset{\mu_G}{\to} G \to \text{Out}\,\pi$ can be realized by a group of homeomorphisms isomorphic to H.

Another interpretation of Theorem 3 is that if G acts topologically and effectively on a closed flat manifold $M(\pi)$, then this group G must be isomorphic to one of the usual finite subgroups of Aff(M), and moreover the extension obtained by lifting to $\mathbf{R}^n$ must be naturally isomorphic to the extension corresponding to a lifting of an affine action.

If one were to begin with an affine action instead of topological action we actually have the strongest possible result.

Theorem 6. *Any effective finite affine action on a closed flat manifold is affinely equivalent to an isometric action on an affinely equivalent manifold.*

Proof. Let $(G, M(\pi))$ be a finite affine action on a closed flat manifold. Then we have an extension $1 \to \pi \to E \to G \to 1$ which is admissible by Lemma 1. Note that $E = \eta^{-1}(G)$ sits inside $A(n)$ since $\pi \subset E(n)$ and $G \subset$ Aff(M). What we are going to do is to show that $h \in A(n)$ in the proof of Theorem 3 can be chosen as that $\tilde{\theta}$: $E \to \tilde{\theta}(E)$ is the identity map not only on π but also on E. (This yields a generalization of the classical Bieberbach theorems.)

Since $C_E(\mathbf{Z}^n) = C_{A(n)}(\mathbf{Z}^n) \cap E = \mathbf{R}^n \cap E$, $C_E(\mathbf{Z}^n)$ is the set of all pure translations in E. Therefore in the exact sequence $1 \to C_E(\mathbf{Z}^n) \to E \overset{\lambda}{\to} E/C_E(\mathbf{Z}^n) = K \to 1$, λ is the restriction of the projection λ: $A(n) \to GL(n, \mathbf{R})$.

We look at the proof for crystallographic groups, and shall see that it also works here. Since $C_E(\mathbf{Z}^n)$ is the maximal abelian, normal subgroup of E, for any θ: $E \overset{\cong}{\to} \theta(E) \subset E(n)$, $\theta(C_E(\mathbf{Z}^n))$ is the maximal abelian, normal subgroup of $\theta(E)$ so that $\theta(C_E(\mathbf{Z}^n) = C_{\theta(E)}(\theta(\mathbf{Z}^n)))$. Therefore this can be thought of as a linear map from $\mathbf{R}^n$ to $\mathbf{R}^n$ and so belongs to $GL(n, \mathbf{R})$, say A. If we choose a basis in terms of $C_E(\mathbf{Z}^n)$, then $\theta(t_v) = t_{A(v)}$, where t_v denotes translation by v.

We must determine what happens to elements of K under θ. We claim that

$$\theta(k, w) = (A, v)(k, w)(A, v)^{-1}$$

for some $v \in \mathbf{R}^n$. This v will be determined by a 1-cocycle f: $K \to \mathbf{R}^n$ which turns out to be principal since $H^1(K; \mathbf{R}^n) = 0$. We do this by trying to determine $\theta(k, w) = (\bar{\theta}(k), ?)$ for some (k, w) in E. Let $(I, a) \in E$. Then $\theta((k, w)(I, a)(k, w)^{-1}) = \theta(I, ka) = (I, Aka)$. On the other hand, if $\theta(k, w) = (B, v)$, then $\theta((k, w)(I, a)(k, w)^{-1}) = (B, v)(I, Aa)(B, v)^{-1} = (I, BAa)$. This implies $B = AkA^{-1}$ so that $\bar{\theta}(k) = AkA^{-1}$. Let us put $\theta(k, w)(AkA^{-1}, Aw + f(k, w))$ for some function $f\colon E \to \mathbf{R}^n$. We want to show that f is independent of w. Suppose $(I, w') \in E$. Then $(I, w')(k, w) = (k, w' + w)$, and this would represent all possible elements of E which map to k. Now $(AkA^{-1}, A(w' + w) + f(k, w' + w)) = \theta(k, w' + w) = \theta((I, w')(k, w)) = \theta(I, w')\theta(k, w) = (AkA^{-1}, Aw' + Aw + f(k, w))$ shows $f(k, w) = f(k, w' + w)$. So f is only a function of k, and we can write $\theta(k, w) = (AkA^{-1}, Aw + f(k))$. Now it is easy to see that f satisfies the 1-cocycle condition: $f(kk') = f(k) = AkA^{-1}f(k')$. This defines a new K-module structure on $\mathbf{R}^n$, namely, $\tau\colon K \to \operatorname{Aut} \mathbf{R}^n$ via $\tau(k) = AkA^{-1}$, and we have $H^1_\tau(K; \mathbf{R}^n) = 0$. So we may write $f(k) = v - (AkA^{-1})v$ for some $v \in \mathbf{R}^n$. Consequently we have

$$\theta(k, w) = \big(AkA^{-1}, Aw + v - AkA^{-1}v\big),$$

which is, of course, the same as $(A, v)(k, w)(A, v)^{-1}$.

Therefore our abstract isomorphism $\theta\colon E \xrightarrow{\cong} \theta(E)$ for $E \subset N_{A(n)}(\pi)$ is realized by an affine diffeomorphism of $M(\pi)$. It is the same affine diffeomorphisms which carries $M(\pi)$ to $M(\theta(\pi))$. q.e.d.

We have seen that affine actions match up with isometric actions extremely nicely. But for topological actions Theorem 3 cannot be strengthened as Theorem 6.

Example. Y. W. Lee [6] constructed a differentiable involution on T^5 with two fixed point components T^3 and $L(j, 1) \# T^3$, where $L(j, 1)$ is a 3-dimensional lens space with j odd. This implies that on the standard flat 5-torus, there is a smooth involution which is not topologically equivalent to any affine involution. If it were topologically equivalent to some affine involution, then it would be topologically equivalent to some isometric one. Then the fixed point set of the original involution should be a homeomorphic image of the fixed point set of the isometric one. But this is impossible because the latter is a geodesic submanifold of T^5.

While we have shown that no "exotic" finite groups can act topologically and effectively on closed flat manifolds, we see that the possible smooth actions can be drastically different from isometric ones. On the other hand, it can be seen from [2, Appendix] that the resulting admissible extensions force the fixed point sets for p-groups to behave cohomologically like isometric actions. This point will be discussed in a subsequent paper.

Combining the Corollary to Lemma 1 with Theorems 3 and 6, (and in the ineffective case using Theorem 4), it is easy to obtain the

Corollary. *The affine equivalences of finite affine G-actions on $M(\pi)$ are in a natural* 1-1 *correspondence with the isomorphism classes of extensions of π by G. (Two extensions E and E', are "isomorphic" if there exists an isomorphism $E \to E'$ which restricts to an automorphism of π.) Moreover, affine actions are topologically equivalent if and only if they are affinely equivalent.*

After this paper was accepted for publication, we discovered that H. Zischang and B. Zimmermann [*Endliche Gruppen von Abbildungsklassen gefaserter* 3-*Mannigfaltigkeiten*, Math. Ann. **240** (1979) 41–62] had earlier obtained Proposition 2 and Theorem 3 when φ: $G \to \mathrm{Out}\, \pi$ is injective.

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